String theory and critical dimensions

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Introduction

The present knowledge of particle physics is summarized by the standard model (SM), which combines the electroweak and strong interactions without real unification. However, this interplay is necessary because some particles take part in both interactions.

The Lagrangian of SM has the structure

$$\mathcal{L} = \frac{1}{4}\,\mathsf{Tr}\,(F_{\mu\nu}F^{\mu\nu}) + \bar{\psi}(i\gamma^{\mu}D_{\mu} - M)\psi + |D\phi|^2 - \lambda(|\phi|^2 - a^2)^2 \ ,$$

with Higgs field ϕ and Yang-Mills type fields

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{n}A_{\mu} + g[A_{\mu}, A_{\nu}] \ , \quad D_{\mu} = \partial_{\mu} - eA_{\mu} \ ,$$

which combine 12 force carriers: 8 gluons, photon and 3 gauge bossons.

Matter fields in SM are represented by:

12 leptons (e, ν_e), (μ, ν_μ), (τ, ν_τ), together with antiparticles,

36 quarks (u, d), (c, s), (t, b), in 3 colors and with antiparticles.

Totally, there are about 60 elementary particles and 20 parameters in SM.

Though SM is in excellent agreement with the experimental data, it is believed that SM is only a step towards the complete fundamental theory.

The main drawback of SM is the absence of the gravity forces in it.

The effects of gravity are quite negligible in particle physics, however they are crucial in cosmology and in the study of the early universe.

At the same time, there is no consistent theory of quantum gravity itself.

Therefore, construction of a quantum theory that would include both gravity and the other forces is fundamentally necessary.

The most promising candidate for the unification of gravity with the other forces is superstring theory.

In this theory all particles, including graviton, are combined in one geometrical object - string and particles are treated as its excited modes.

Since all particles arise from string excitation, all particles and all forces are naturally incorporated into a single theory.

The fundamental theory should have a certain degree of uniqueness.
A theory with adjustable dimensionless parameters, like SM, is not unique.

String theory has only one dimensionful parameter - string length.
It is the first sign that string theory is rather unique.

Another sign of the uniqueness of string theory is the fact that it fixes the spacetime dimension by symmetry principles.

Namely, the quantum realization of spacetime symmetries fixes the spacetime dimension, which is 26 for bosonic strings and it is 10 for superstrings. This numbers are called critical dimensions.

Quantization of a bosonic string and a realization of the critical dimension by a new method is the main topic of the seminar.

We use the static gauge approach, which is most natural from the physical point of view, however it creates hard operator ordering ambiguities.

We show how one can avoid the quantization problems and how to calculate the critical dimension in a simple way.

Particle dynamics

The geometric action

$$S = -m \int \mathrm{d}s = -M \int \mathrm{d}\tau \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} \ .$$

The Polyakov action

$$S = \int \mathrm{d}\tau \left[\frac{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}{2\lambda} - \frac{\lambda\, M^2}{2} \right] \ .$$

The Dirac action

$$S = \int \mathrm{d}\tau \left[p_\mu \dot{x}^\mu - \frac{\lambda}{2} \left(g^{\mu\nu}\, p_\mu p_\nu + M^2 \right) \right] \ .$$

Connection between the velocity and momentum

$$\dot{x}^\mu = \lambda g^{\mu\nu}\, p_\nu \ .$$

The mass-shell condition

$$g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + \lambda^2\, M^2 = 0 \ .$$

String dynamics

The Nambu-Goto action

$$S = -\frac{1}{2\pi} \int \mathrm{d}\tau \mathrm{d}\sigma \sqrt{(\dot{x} \cdot x')^2 - (\dot{x})^2 \, (x')^2} \ .$$

The Polyakov action

$$S = -\frac{1}{4\pi} \int \mathrm{d}\tau \mathrm{d}\sigma \sqrt{h} \ h^{\alpha\beta} \left(\partial_a x \cdot \partial_\beta x\right) \ .$$

The Dirac action

$$S = \frac{1}{2\pi} \int \mathrm{d}\tau \int \mathrm{d}\sigma \left[(p \cdot \dot{x}) - \lambda_1 (p^2 + (x')^2) - \lambda_2 (p \cdot x') \right] \ .$$

Constraints

$$p^2 + (x')^2 = 0 \ , \qquad p \cdot x' = 0 \ .$$

Particle in a static space-time

The action of a particle in the first order formulation

$$S = \int \mathrm{d}\tau \Big[p_\mu \, \dot{x}^\mu - \frac{\lambda}{2} \left(g^{\mu\nu} p_\mu p_\nu + M^2 \right) \Big] \; .$$

The mass shell condition

$$g^{\mu\nu} p_\mu p_\nu + M^2 = 0 \; .$$

The static gauge $\boxed{x^0 + p_{_0} \tau = 0 \; .}$

The reduced action

$$S = \int \mathsf{d}\tau \Big[p_n \, \dot{x}^n - \frac{1}{2} \, (p_{_0})^2 \Big] \; .$$

$-p_0 = E > 0$ is the particle energy.

Static space-time metric tensor

$$g_{\mu\nu} = \begin{pmatrix} g_{00}(x) & 0 \\ 0 & g_{mn}(x) \end{pmatrix}, \qquad g_{00} < 0 \,.$$

Space-time coordinates $x^\mu = (x^0, x)$.

Time component $\quad g_{00}(x) = -e^{f(x)}$.

The squared energy

$$E^2 = h^{mn}(x)\, p_m p_n + M^2\, e^{f(x)} \,.$$

This function is associated with the Hamiltonian of a non-relativistic particle moving in the potential $M^2\, e^{f(x)}$ in a curved background with metric tensor

$$h_{mn} = e^{-f(x)} g_{mn}(x) \,,$$

and $h^{mn}(x)$ is the inverse metric.

Particle quantization in static gauge

Scalar product with a covariant measure

$$\langle\psi_2|\psi_1\rangle = \int \mathsf{d}^N x \sqrt{h(x)}\, \psi_2^*(x)\, \psi_1(x) \ ,$$

where $h(x) = \det h_{mn}(x)$.

The momentum operators $p_n = -i\partial_n - \frac{i}{4}\partial_n \log h$.

The energy square operator

$$E^2 = -\Delta_h + a\mathcal{R}_h(x) + M^2\, e^{f(x)} \ ,$$

where Δ_h is the Laplace-Beltrami operator and $\mathcal{R}_h$ is the scalar curvature.

How to fix a? [DeWitt, Bastianeli,...]

String dynamics in static gauge

Open string action in the first order formulation

$$S=\int \mathrm{d}\tau \int_0^\pi \frac{\mathrm{d}\sigma}{\pi}\Big(\mathcal{P}_\mu \dot{X}^\mu - \lambda_1(\mathcal{P}_\mu\mathcal{P}^\mu + X'_\mu X'^\mu) - \lambda_2(\mathcal{P}_\mu X'^\mu)\Big) \ .$$

Virasoro constraints $\quad \mathcal{P}_\mu\mathcal{P}^\mu + X'_\mu X'^\mu = 0 \ , \qquad \mathcal{P}_\mu X'^\mu = 0 \ .$

Static gauge $\quad X^0 + \mathcal{P}_0\tau = 0 \ , \qquad \mathcal{P}'_0 = 0 \ .$

The reduced action

$$S=\int \mathrm{d}\tau \int_0^\pi \frac{\mathrm{d}\sigma}{\pi}\left(\mathcal{P}_k \dot{X}^k - \frac{1}{2}\,\mathcal{P}_0^2\right) \ .$$

Free-field Hamiltonian

$$H=\frac{1}{2}\int_0^\pi \frac{\mathrm{d}\sigma}{\pi}\,\left(\vec{\mathcal{P}}^2 + \vec{X}'^{\,2}\right) \ .$$

The free fields on the (τ, σ) strip

$$X^k(\tau,\sigma) = \frac{1}{2}\phi^k(\tau+\sigma) + \frac{1}{2}\phi^k(\tau-\sigma) \ .$$

Mode expansion $\qquad \phi^k(z) = q^k + p^k z + \mathrm{i}\sum_{n\neq 0} \frac{a_n^k}{n}\ \mathrm{e}^{-\mathrm{i}nz} \ .$

Canonical Poisson brackets

$$\{p^k, q^l\} = \delta^{kl} \ , \qquad \{a_m^k, a_n^l\} = \mathrm{i}m\,\delta^{kl}\,\delta_{m+n} \ .$$

The generators of conformal transformations

$$L_m = \frac{1}{2}\int_0^{2\pi} \frac{\mathrm{d}z}{2\pi}\ \mathrm{e}^{\mathrm{i}mz}\,\vec{\phi}'(z)^2 = \frac{1}{2}\sum_{n=-\infty}^{\infty} \vec{a}_{m-n}\vec{a}_n \ ,$$

provide the Witt algebra $\qquad \{L_m, L_n\} = \mathrm{i}\,(m-n)\,L_{m+n} \ .$

The constraints read

$$L_m = 0, \qquad m \neq 0 \ .$$

The dynamical integrals

$$P^\mu = \int_0^\pi \frac{\mathrm{d}\sigma}{\pi}\,\mathcal{P}^\mu\ , \qquad J^{\mu\nu} = \int_0^\pi \frac{\mathrm{d}\sigma}{\pi}\ \left(\mathcal{P}^\mu\, X^\nu - \mathcal{P}^\nu\, X^\mu\right)\ .$$

Reduced dynamical integrals

$$P^k = p^k\ , \qquad J^{kl} = p^k q^l - p^k q^l + \mathrm{i}\sum_{n\neq 0}\frac{a^k_{-n}a^l_n}{n}\ ,$$

$$P^0 = p^0 = \sqrt{2L_0}\ , \qquad J^{0k} = p^0 q^k + \frac{\mathrm{i}}{p^0}\sum_{n\neq 0}\frac{a^k_n}{n}\ L_{-n}\ .$$

Deformed boosts

$$\mathcal{J}^{0k} = J^{0k} + \frac{\mathrm{i}}{p^0}\sum_{j\geq 2}\left(\sum_{n_1,\dots,n_j} f_j(p^0)\ \frac{a^k_n}{n}\ L_{-n_1}\dots L_{-n_j}\right)\ .$$

Poisson brackets of the form

$$\{\mathcal{J}^{0k},\, L_m\} = \mathcal{A}^k_m\, L_m\ .$$

String quantization in static gauge

The ground state

$$a_0^k|\vec{p}\rangle = p^k|\vec{p}\rangle \ , \qquad a_n^k|\vec{p}\rangle = 0 \ , \qquad n > 0 \ .$$

Virasoro generators $L_0 = \frac{1}{2}\,\vec{p}^2 + N, \qquad N = \sum_{n>0} \vec{a}_{-n}\,\vec{a}_n \ .$

The Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \tfrac{D-1}{12}(m^3 - m)\delta_{m+n} \ .$$

The physical states $\qquad L_1|\Psi\rangle = 0 \ , \qquad L_2|\Psi\rangle = 0 \ .$

The energy operator $\qquad p^0 = \sqrt{\vec{p}^{\,2} + 2(N-a)} \ .$

The boost operators

$$\mathcal{J}^{0k} = p^0 q^k + \frac{\mathrm{i}}{p^0} \sum_{n=1}^{N} \sum_{(n_1,\ldots,n_j)} f^{(n_1,..,n_j)}(p^0) L_{-n_1} \cdots L_{-n_j} \frac{a_n^k}{n} \ .$$

The action of boosts on the first excited level

$$\mathcal{J}^{0k}|\vec{p},\,1\rangle = \left(q^k p^0 - \frac{\mathrm{i}\,p^k}{2p^0} + \frac{\mathrm{i}\,f^{(1)}}{p^0}\,L_{-1}a_1^k\right)|\vec{p},\,1\rangle\ .$$

Phys. conditions $L_1\mathcal{J}^{0k}|\vec{p},\,1\rangle = 0\ ,\qquad [\mathcal{J}^{0k},\mathcal{J}^{0l}]|\vec{p},\,1\rangle = \mathrm{i}\,|\vec{p},\,1\rangle\,.$

The solution: $\qquad f^{(1)} = 1\ ,\qquad a = 1\,.$

The second level calculations

$$\left(q^k p^0 - \frac{\mathrm{i}\,p^k}{2p^0} + \frac{\mathrm{i}}{p^0}\,L_{-1}a_1^k + \frac{\mathrm{i}\,f^{(2)}}{2\,p^0}\,L_{-2}a_2^k + \frac{\mathrm{i}\,f^{(1,1)}}{2\,p^0}L_{-1}L_{-1}a_2^k\right)|\vec{p},\,2\rangle\ ,$$

with conditions $L_1\mathcal{J}^{0k}|\vec{p},\,2\rangle = 0,\ \ L_2\mathcal{J}^{0k}|\vec{p},\,2\rangle = 0$, yield

$$f^{(1,1)} = -\frac{1}{e+1}\ ,\qquad f^{(2)} = \frac{e}{e+1}\ ,\quad e = 2(p^0)^2\ ,\quad D = 26\ .$$

Connection to the covariant quantization

The physical states in covariant quantization

$$\hat{L}_m||\psi_{ph}\rangle = 0 \ , \quad \text{for} \quad m > 0, \qquad \text{and} \quad (\hat{L}_0 - 1)||\psi_{ph}\rangle = 0 \ ,$$

$$\hat{L}_m = L_m - L^0_m \ , \qquad \text{with} \qquad L^0_m = \frac{1}{2}\, a^0_{m-n}\, a^0_n \ .$$

The physical states of static gauge quantization become physical states of the covariant quantization.

The boost operators in the covariant quantization

$$J^{0k} = p^0 q^k - p^k q^0 + \mathrm{i} \sum_{n>0} \left(\frac{a^0_{-n} a^k_n}{n} - \frac{a^k_{-n} a^0_n}{n} \right)$$

have no ordering ambiguity. From the expansion

$$a^0_{-n}||p^0, \vec{p};\, 0, N\rangle = \sum_{(n_1,\ldots,n_j)} \tilde{f}^{(n_1,\ldots,n_j)}(p^0)\ L^0_{-n_j} \cdots L^0_{-n_1}\, ||p^0, \vec{p};\, 0, N\rangle \ ,$$

follows $\quad f^{(n_1,\ldots,n_j)}(p^0) = p^0\, \tilde{f}^{(n_1,\ldots,n_j)}(p^0).$

Summary

We have discussed the quantization of a particle in a static spacetimes in the coordinate representation, which corresponds to the static gauge.

The key point of this analysis is the construction of the squared energy operator which has the form of a non-relativistic particle Hamiltonian in a curved space.

The description of a quantum particle in terms of position dependent wave functions seems to be most natural.

But in order to respect the relativistic principles, the generators for transformations involving time (energy, boosts) become non-local.

We have proposed a new treatment of the bosonic strings in static gauge.

It has been shown that the string dynamics in D dimensional Minkowski space can be described by $D-1$ component conformal free-field theory, restricted by the constraints on the Virasoro generators $L_m = 0$, $m \neq 0$.

The structure of the boost operators has been found on the basis of classical calculations.

This structure defines the boosts up to some energy dependent coefficients.

These coefficients can be calculated to any desirable level, but their closed form is still missing.

Simple low level calculations of the boost operators commutators define the string mass spectrum and the space-time critical dimension.

We have shown the equivalence between the static gauge and the covariant quantization.

This equivalence shows that the coefficients, we were looking for in the static gauge quantization, are just the expansion coefficients of the oscillator excitations in terms of the Virasoro excitations.